Notes on Elastocapillary Thinning of a Viscoelastic Filament

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Abstract

A thin liquid filament (a capillary bridge) thins because surface tension creates a capillary pressure that drives extensional flow. For viscoelastic liquids, the resulting polymeric tensile stress can balance capillarity and leads to the celebrated *elastocapillary* regime: the filament radius decays exponentially and the extensional strain-rate becomes approximately constant. This document revisits the derivation of the radius evolution $R(t)$ and explains why the boundary conditions imply a Weissenberg number $\mathrm{Wi} = \lambda\dot{\varepsilon} = 2/3$ in the elastocapillary regime. Unlike the original derivation (which uses multi-mode FENE dumbbells), we use the *Upper-Convected Maxwell* (UCM) constitutive equation written directly in terms of *modulus* G, *relaxation time* λ, and *viscosities* (no conformation tensors).

Notation and what we are solving for

We study a long, nearly uniform cylindrical filament of radius $R(t)$ (time-dependent). The key quantity measured in capillary-thinning rheometry is the radius evolution $R(t)$. A central kinematic quantity is the (uniaxial) extensional strain-rate

$$\dot{\varepsilon}(t) \equiv \frac{\partial v_z}{\partial z} \quad (\text{units: s}^{-1}).$$

The material parameters are summarized in the following table.

Table 1: Material parameters

Symbol	Meaning	Unit
Γ	(Liquid-vapour) surface tension	N/m
η_s	Solvent (Newtonian) viscosity	$\mathrm{Pa}\cdot\mathrm{s}$
G	Elastic modulus (UCM polymer parameter)	Pa
λ	Relaxation time (UCM polymer parameter)	s
η_p	Polymer viscosity for a Maxwell element, $\eta_p = G\lambda$	$\mathrm{Pa}\cdot\mathrm{s}$

The Weissenberg number in extension is

$$\mathrm{Wi} \equiv \lambda\dot{\varepsilon}.$$

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What we will derive (using single-mode UCM). In the elastocapillary regime:

$$R(t) = R_E \exp\left(-\frac{t - t_E}{3\lambda}\right), \qquad \dot{\varepsilon} = \frac{2}{3\lambda}, \qquad \mathrm{Wi} = \frac{2}{3}.$$

The constants t_E and R_E mark the onset of the elastocapillary regime (set by early-time transients). The *slope* $-1/(3\lambda)$ is the main rheometric signature.[1]

1 Physical and mathematical assumptions (the "0D uniform filament" model)

This derivation follows the same *modeling level* as the uniform-cylinder analysis used by Entov & Hinch (1997): once a filament is formed, there is typically a long central region that is approximately cylindrical, and its dynamics can be approximated as *spatially uniform* (depends only on time).

Assumptions

1. **Axisymmetric, slender filament with a uniform cylindrical core.** In the core region, the radius is approximately constant along z at each time: $R = R(t)$.

2. **Incompressible liquid.** Density is constant and $\nabla \cdot \boldsymbol{v} = 0$.

3. **Negligible inertia and gravity in the regime of interest.** So the momentum balance reduces to a quasi-static stress balance (creeping/slender regime). (These are standard in capillary-thinning analyses at small Reynolds number.)

4. **Constant surface tension Γ.** No surfactant dynamics.

5. **Constitutive law: UCM polymer + Newtonian solvent.**[2] Total stress:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\eta_s \boldsymbol{D} + \boldsymbol{\tau},$$

   where $\boldsymbol{D} = \frac{1}{2}\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}}\right]$[3] and $\boldsymbol{\tau}$ is the polymer extra stress satisfying the UCM equation (Section 4).

2 Kinematics: link between radius and extensional strain-rate

We assume the flow in the uniform filament core is *uniaxial extension*: the filament stretches along z while shrinking in the transverse directions.

[1] More specifically,

$$\frac{R_{min}(t)}{R_0} = \left(\frac{GR_0}{2\Gamma}\right)^{1/3} \exp\left[-\frac{(t - t^*)}{3\lambda_E}\right], \qquad \text{for } t \geq t^*,$$

where R_0 is the initial radius of the filament, t^* is the onset moment of the elastocapillary regime, λ_E specifies the extensional relaxation time.

[2] In the original derivation, the authors used a multi-mode FENE-CR model to study the effect of a spectrum of relaxation times as well as late-time finite extensibility effect.

[3] There are two common definitions of the rate-of-strain tensor, one is $\boldsymbol{D} = \frac{1}{2}\left[\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}}\right]$, while the other is $\dot{\boldsymbol{\gamma}} = 2\boldsymbol{D} = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}}$.

2.1 Choose a velocity field consistent with uniaxial extension

A standard spatially linear velocity field for uniaxial extension is (in Cartesian coordinates):

$$v_x = -\frac{\dot{\varepsilon}(t)}{2}\, x, \qquad v_y = -\frac{\dot{\varepsilon}(t)}{2}\, y, \qquad v_z = \dot{\varepsilon}(t)\, z.$$

This represents stretching along z and equal compression in x and y.[4]

2.2 Check incompressibility explicitly

Compute the divergence:

$$\nabla \cdot \boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \left(-\frac{\dot{\varepsilon}}{2}\right) + \left(-\frac{\dot{\varepsilon}}{2}\right) + (\dot{\varepsilon}) = 0.$$

So the field is incompressible.

2.3 Kinematic boundary condition at the free surface

Let the filament free surface be the material surface

$$x^2 + y^2 = R(t)^2 \quad \text{or} \quad r = R(t),$$

which correspond to Cartesian and cylindrical coordinates, respectively. The kinematic condition is: *the surface moves with the fluid.* In axisymmetry this reduces to:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = v_r\big|_{r=R(t)}.$$

In the Cartesian representation, the radial velocity is the component in the transverse direction. For a point on the surface with transverse radius $r = \sqrt{x^2 + y^2} = R(t)$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\frac{\dot{\varepsilon}(t)}{2}\, r.$$

Setting $r = R(t)$ gives

$$\boxed{\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{\dot{\varepsilon}(t)}{2}\, R(t).}$$

This is the fundamental kinematic link between radius and extensional strain-rate.

3 Stress boundary conditions and the key 1D stress balance

The central step is to convert capillarity into a relation between the capillary pressure Γ/R and the *tensile stress difference* inside the filament.

[4] The velocity field is expressed in Cartesian coordinates as

$$\boldsymbol{v} = -\tfrac{1}{2}\dot{\varepsilon}x\hat{\boldsymbol{x}} - \tfrac{1}{2}\dot{\varepsilon}y\hat{\boldsymbol{y}} + \dot{\varepsilon}z\hat{\boldsymbol{z}}.$$

Alternatively, the velocity field can be expressed in cylindrical coordinates as

$$\boldsymbol{v} = -\tfrac{1}{2}\dot{\varepsilon}r\hat{\boldsymbol{r}} + \dot{\varepsilon}z\hat{\boldsymbol{z}}.$$

Here, the hatted vectors are basis vectors of the corresponding directions.

3.1 Normal stress balance at a cylindrical free surface

At the liquid–air interface, the normal stress jump equals surface tension times curvature (Young–Laplace law in stress form):

$$\boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\Big|_{\text{inside}} - \boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\Big|_{\text{outside}} = -\Gamma\kappa,$$

where κ is the mean curvature (sign conventions vary; we will use a common "inside minus outside" form). For a long cylinder of radius R, the principal curvatures are:

$$\kappa_1 = \frac{1}{R}, \qquad \kappa_2 = 0 \quad \Rightarrow \quad \kappa = \kappa_1 + \kappa_2 = \frac{1}{R}.$$

Assume outside is air at (gauge) pressure 0 with negligible viscous stress, so $\boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n}\big|_{\text{outside}} = 0$. The normal vector is radial; thus $\boldsymbol{n} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{n} = \sigma_{rr}$. So the boundary condition becomes

$$\boxed{\sigma_{rr}(r = R(t)) = -\frac{\Gamma}{R(t)}.}$$

3.2 Axial traction condition (the second boundary condition)

Entov & Hinch (1997) use the approximation that the filament is attached to large, nearly stagnant end drops so that the axial normal stress in the uniform filament core is (approximately) zero:

$$\boxed{\sigma_{zz} \approx 0.}$$

This is an *idealized* but very common closure in the "uniform cylinder" model: it allows one to eliminate the pressure and obtain a simple relation between capillarity and the *extensional stress difference*. (If σ_{zz} were a nonzero constant, it would add an additive constant to the balance; when Γ/R becomes large at small R, the dominant behavior is unchanged.)

3.3 Eliminate the pressure to get the stress-difference balance

Write the Cauchy stress in the filament:

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\eta_s\boldsymbol{D} + \boldsymbol{\tau}.$$

For uniaxial extension, the rate-of-deformation tensor $\boldsymbol{D}$ is diagonal with

$$D_{zz} = \dot{\varepsilon}, \qquad D_{rr} = D_{\theta\theta} = -\frac{\dot{\varepsilon}}{2}.$$

Therefore the solvent (Newtonian) contribution to normal stresses is:

$$(2\eta_s D)_{zz} = 2\eta_s\dot{\varepsilon}, \qquad (2\eta_s D)_{rr} = 2\eta_s\left(-\frac{\dot{\varepsilon}}{2}\right) = -\eta_s\dot{\varepsilon}.$$

Thus

$$\sigma_{zz} = -p + 2\eta_s\dot{\varepsilon} + \tau_{zz}, \qquad \sigma_{rr} = -p - \eta_s\dot{\varepsilon} + \tau_{rr}.$$

Now apply the two boundary conditions:

$$\sigma_{zz} = 0, \qquad \sigma_{rr} = -\frac{\Gamma}{R}.$$

Subtract the second from the first:

$$\sigma_{zz} - \sigma_{rr} = 0 - \left(-\frac{\Gamma}{R}\right) = \frac{\Gamma}{R}.$$

But also, from the formulae above,

$$\begin{aligned}\sigma_{zz} - \sigma_{rr} &= (-p + 2\eta_s\dot{\varepsilon} + \tau_{zz}) - (-p - \eta_s\dot{\varepsilon} + \tau_{rr}) \\ &= 3\eta_s\dot{\varepsilon} + (\tau_{zz} - \tau_{rr}).\end{aligned}$$

Therefore the key 1D balance is

$$\boxed{\frac{\Gamma}{R(t)} = 3\eta_s\dot{\varepsilon}(t) + \big(\tau_{zz}(t) - \tau_{rr}(t)\big).}$$

4 Constitutive law: Upper-Convected Maxwell in stress form

4.1 UCM equation (single Maxwell mode)

The UCM model for the polymer extra stress $\boldsymbol{\tau}$ is:

$$\boxed{\boldsymbol{\tau} + \lambda\overset{\nabla}{\boldsymbol{\tau}} = 2\eta_p\boldsymbol{D},} \qquad \eta_p = G\lambda.$$

Here $\overset{\nabla}{\boldsymbol{\tau}}$ is the *upper-convected derivative*:

$$\overset{\nabla}{\boldsymbol{\tau}} = \frac{\partial\boldsymbol{\tau}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{\tau} - \left[(\nabla\boldsymbol{v})^{\mathrm{T}}\cdot\boldsymbol{\tau} + \boldsymbol{\tau}\cdot(\nabla\boldsymbol{v})\right].$$

Uniform filament simplification. In the uniform core region we assume $\boldsymbol{\tau}$ is spatially uniform, so $\boldsymbol{v}\cdot\nabla\boldsymbol{\tau} = 0$.[5]

4.2 Compute velocity gradient and rate-of-strain tensor explicitly

From the velocity field

$$v_x = -\frac{\dot{\varepsilon}}{2}x, \quad v_y = -\frac{\dot{\varepsilon}}{2}y, \quad v_z = \dot{\varepsilon}z,$$

the velocity gradient matrix is diagonal:

$$\nabla\boldsymbol{v} = \begin{pmatrix} -\dot{\varepsilon}/2 & 0 & 0 \\ 0 & -\dot{\varepsilon}/2 & 0 \\ 0 & 0 & \dot{\varepsilon} \end{pmatrix}.$$

Because this matrix is symmetric (diagonal), $\boldsymbol{D} = \nabla\boldsymbol{v}$.[6]

[5]In fact, we affirm that $\boldsymbol{\tau} = \boldsymbol{\tau}(t)$.

[6]The alternative approach is to use cylindrical coordinates. From the velocity field

$$v_r = -\tfrac{\dot{\varepsilon}}{2}r, \quad v_z = \dot{\varepsilon}z,$$

the volocity gradient tensor is also diagonal in cylindrical coordinates:

$$\nabla\boldsymbol{v} = -\tfrac{\dot{\varepsilon}}{2}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} - \tfrac{\dot{\varepsilon}}{2}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \dot{\varepsilon}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}.$$

4.3 Upper-convected derivative for diagonal stresses

Assume (consistent with symmetry) that $\boldsymbol{\tau}$ is also diagonal in this flow:

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & 0 & 0 \\ 0 & \tau_{yy} & 0 \\ 0 & 0 & \tau_{zz} \end{pmatrix}, \qquad \tau_{xx} = \tau_{yy} = \tau_{rr}.$$

Here, the components of $\boldsymbol{\tau}$ are expressed in Cartesian coordinates.[7]

Using

$$\overset{\triangledown}{\boldsymbol{\tau}} = \dot{\boldsymbol{\tau}} - (\nabla \boldsymbol{v})^{\mathrm{T}} \cdot \boldsymbol{\tau} - \boldsymbol{\tau} \cdot (\nabla \boldsymbol{v}),$$

and the fact that everything is diagonal, the products are easy:

$$\left((\nabla \boldsymbol{v})^{\mathrm{T}} \cdot \boldsymbol{\tau}\right)_{ii} = (\nabla \boldsymbol{v})_{ii} \tau_{ii}, \qquad \left(\boldsymbol{\tau} \cdot (\nabla \boldsymbol{v})\right)_{ii} = \tau_{ii} (\nabla \boldsymbol{v})_{ii}.$$

So

$$\overset{\triangledown}{\tau}_{ii} = \dot{\tau}_{ii} - 2(\nabla \boldsymbol{v})_{ii} \tau_{ii}.$$

Therefore:

$$\overset{\triangledown}{\tau}_{zz} = \dot{\tau}_{zz} - 2\dot{\varepsilon}\, \tau_{zz},$$

because $(\nabla \boldsymbol{v})_{zz} = \dot{\varepsilon}$, and

$$\overset{\triangledown}{\tau}_{rr} = \dot{\tau}_{rr} - 2\left(-\frac{\dot{\varepsilon}}{2}\right) \tau_{rr} = \dot{\tau}_{rr} + \dot{\varepsilon}\, \tau_{rr}.$$

4.4 Component-wise UCM equations

Plugging into $\boldsymbol{\tau} + \lambda \overset{\triangledown}{\boldsymbol{\tau}} = 2\eta_p \boldsymbol{D}$:

Axial component (zz).

$$\tau_{zz} + \lambda(\dot{\tau}_{zz} - 2\dot{\varepsilon}\tau_{zz}) = 2\eta_p D_{zz} = 2\eta_p \dot{\varepsilon}.$$

Equivalently,

$$\boxed{\dot{\tau}_{zz} = 2\dot{\varepsilon}\tau_{zz} - \frac{1}{\lambda}\tau_{zz} + \frac{2\eta_p}{\lambda}\dot{\varepsilon}.}$$

Radial/transverse component (rr).

$$\tau_{rr} + \lambda(\dot{\tau}_{rr} + \dot{\varepsilon}\tau_{rr}) = 2\eta_p D_{rr} = 2\eta_p \left(-\frac{\dot{\varepsilon}}{2}\right) = -\eta_p \dot{\varepsilon}.$$

Equivalently,

$$\boxed{\dot{\tau}_{rr} = -\dot{\varepsilon}\tau_{rr} - \frac{1}{\lambda}\tau_{rr} - \frac{\eta_p}{\lambda}\dot{\varepsilon}.}$$

These are ordinary differential equations (ODEs) in time.

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[7] Similarly, in cylindrical coordinates,

$$\boldsymbol{\tau} = \tau_{rr}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \tau_{\theta\theta}\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \tau_{zz}\hat{\boldsymbol{z}}\hat{\boldsymbol{z}},$$

where $\tau_{rr} = \tau_{\theta\theta}$.

5 Early-time viscous thinning (Newtonian-like) and its radius law

At very early times (or if the polymer stress is initially small), the polymer contribution to the stress balance is negligible:

$$\tau_{zz} - \tau_{rr} \approx 0.$$

Then the key balance reduces to

$$\frac{\Gamma}{R} \approx 3\eta_s \dot{\varepsilon}.$$

So

$$\dot{\varepsilon} \approx \frac{\Gamma}{3\eta_s R}.$$

Insert this into the kinematic relation $\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{1}{2}\dot{\varepsilon}R$:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{1}{2}\left(\frac{\Gamma}{3\eta_s R}\right) R = -\frac{\Gamma}{6\eta_s}.$$

This is constant, so integrate:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{\Gamma}{6\eta_s} \;\Rightarrow\; \mathrm{d}R = -\frac{\Gamma}{6\eta_s}\,\mathrm{d}t \;\Rightarrow\; \int_{R_0}^{R(t)} \mathrm{d}R = -\frac{\Gamma}{6\eta_s}\int_0^t \mathrm{d}t,$$

giving

$$R(t) - R_0 = -\frac{\Gamma}{6\eta_s} t \quad \Rightarrow \quad \boxed{R(t) = R_0 - \frac{\Gamma}{6\eta_s}\, t.}$$

This is the same "linear in time" viscous thinning law discussed in Entov & Hinch (1997).

When does this end? It ends when polymeric tensile stress grows to become comparable to Γ/R. In many experiments this transition is very fast and may occur before the first recorded data point.

6 Elastocapillary thinning for a single-mode UCM

The elastocapillary regime is characterized by:

1. **Polymer dominates the stress balance:**

   $$\tau_{zz} - \tau_{rr} \gg 3\eta_s \dot{\varepsilon}.$$

   So

   $$\boxed{\tau_{zz} - \tau_{rr} \approx \frac{\Gamma}{R}.}$$

2. **Large tensile stress compared to the modulus scale:**

   $$\tau_{zz} \sim \frac{\Gamma}{R} \gg G.$$

   Since $\eta_p = G\lambda$ and $\dot{\varepsilon}$ will turn out to be $\mathcal{O}(1/\lambda)$, the "forcing" terms $\sim \eta_p \dot{\varepsilon}/\lambda \sim G\dot{\varepsilon}$ are $\mathcal{O}(G)$ and thus small compared to τ_{zz} when $\Gamma/R \gg G$.

6.1 Simplify the axial UCM equation in the large-stress limit

Start with the exact axial ODE:

$$\dot{\tau}_{zz} = 2\dot{\varepsilon}\tau_{zz} - \frac{1}{\lambda}\tau_{zz} + \frac{2\eta_p}{\lambda}\dot{\varepsilon}.$$

When $\tau_{zz} \gg G$, the last term (order $G\dot{\varepsilon}$) is negligible versus τ_{zz}/λ. So we approximate:

$$\boxed{\dot{\tau}_{zz} \approx \left(2\dot{\varepsilon} - \frac{1}{\lambda}\right)\tau_{zz}.}$$

Also, in strong uniaxial extension, the transverse stress τ_{rr} remains $\mathcal{O}(G)$ and is much smaller than τ_{zz}, so

$$\tau_{zz} - \tau_{rr} \approx \tau_{zz}.$$

Therefore, in the elastocapillary regime,

$$\boxed{\tau_{zz} \approx \frac{\Gamma}{R}.}$$

6.2 Derive the constant strain-rate and Wi = 2/3

We now have two key relations:

$$\tau_{zz} \approx \frac{\Gamma}{R} \qquad \text{and} \qquad \dot{\tau}_{zz} \approx \left(2\dot{\varepsilon} - \frac{1}{\lambda}\right)\tau_{zz}.$$

Step 1: Differentiate the capillary balance $\tau_{zz} = \Gamma/R$.

$$\tau_{zz} = \frac{\Gamma}{R} \;\Rightarrow\; \dot{\tau}_{zz} = \Gamma\frac{\mathrm{d}}{\mathrm{d}t}(R^{-1}) = \Gamma(-1)R^{-2}\dot{R} = -\frac{\Gamma}{R^2}\dot{R}.$$

Step 2: Divide by $\tau_{zz} = \Gamma/R$ to get the relative growth rate.

$$\frac{\dot{\tau}_{zz}}{\tau_{zz}} = \frac{-\Gamma R^{-2}\dot{R}}{\Gamma R^{-1}} = -\frac{\dot{R}}{R}.$$

Step 3: Use the kinematic relation $\dot{R} = -(\dot{\varepsilon}/2)R$.

$$-\frac{\dot{R}}{R} = -\frac{-(\dot{\varepsilon}/2)R}{R} = \frac{\dot{\varepsilon}}{2}.$$

So we have shown

$$\boxed{\frac{\dot{\tau}_{zz}}{\tau_{zz}} = \frac{\dot{\varepsilon}}{2}.}$$

Interpretation: because $\tau_{zz} \sim \Gamma/R$ and R shrinks, the required tensile stress *must increase* at rate $\dot{\varepsilon}/2$.

Step 4: Compute $\dot{\tau}_{zz}/\tau_{zz}$ from the Maxwell relaxation/stretching equation. From

$$\dot{\tau}_{zz} \approx \left(2\dot{\varepsilon} - \frac{1}{\lambda}\right)\tau_{zz},$$

divide by τ_{zz}:

$$\boxed{\frac{\dot{\tau}_{zz}}{\tau_{zz}} \approx 2\dot{\varepsilon} - \frac{1}{\lambda}.}$$

Step 5: Equate the two expressions for $\dot{\tau}_{zz}/\tau_{zz}$.

$$\frac{\dot{\varepsilon}}{2} = 2\dot{\varepsilon} - \frac{1}{\lambda}.$$

Solve step-by-step:

$$\frac{\dot{\varepsilon}}{2} - 2\dot{\varepsilon} = -\frac{1}{\lambda} \;\Rightarrow\; \left(\frac{1}{2} - 2\right)\dot{\varepsilon} = -\frac{1}{\lambda} \;\Rightarrow\; \left(-\frac{3}{2}\right)\dot{\varepsilon} = -\frac{1}{\lambda}.$$

Multiply both sides by $-2/3$:

$$\dot{\varepsilon} = \frac{2}{3}\frac{1}{\lambda}.$$

So

$$\boxed{\dot{\varepsilon} = \frac{2}{3\lambda}} \qquad \Rightarrow \qquad \boxed{\mathrm{Wi} = \lambda\dot{\varepsilon} = \frac{2}{3}.}$$

6.3 Derive the exponential radius law

Use $\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{1}{2}\dot{\varepsilon}R$ with constant $\dot{\varepsilon} = 2/(3\lambda)$:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{1}{2}\left(\frac{2}{3\lambda}\right)R = -\frac{1}{3\lambda}R.$$

Separate variables:

$$\frac{\mathrm{d}R}{R} = -\frac{1}{3\lambda}\,\mathrm{d}t.$$

Integrate from elastocapillary onset time t_E (radius R_E) to time t:

$$\int_{R_E}^{R(t)} \frac{\mathrm{d}R}{R} = -\frac{1}{3\lambda}\int_{t_E}^{t} \mathrm{d}t.$$

Compute integrals:

$$\ln\left(\frac{R(t)}{R_E}\right) = -\frac{t - t_E}{3\lambda}.$$

Exponentiate:

$$\boxed{R(t) = R_E \exp\left(-\frac{t - t_E}{3\lambda}\right).}$$

This is the central elastocapillary thinning result.[8]

[8] This result for the time evolution of filament radius can be derived with a different approach. See Appendix for more information.

6.4 Why 2/3 and not 1/2? (the boundary-condition logic)

A useful comparison:

- If the tensile stress only needed to be *constant* in time, then we would require $\dot{\tau}_{zz} = 0$ in the simplified Maxwell law $\dot{\tau}_{zz} = (2\dot{\varepsilon} - 1/\lambda)\tau_{zz}$. Setting $\dot{\tau}_{zz} = 0$ would give $2\dot{\varepsilon} = 1/\lambda$, i.e.

$$\dot{\varepsilon} = \frac{1}{2\lambda}.$$

- In a thinning filament, however, the capillary pressure is *not constant*: Γ/R increases as R decreases. The boundary condition $\sigma_{rr} = -\Gamma/R$ combined with $\sigma_{zz} \approx 0$ forces the tensile stress difference to scale like Γ/R, so it must grow in time. That required growth rate is exactly $\dot{\varepsilon}/2$ (derived above), and adding this requirement shifts $1/(2\lambda)$ up to $2/(3\lambda)$: the extra $+1/6$ (in Wi units) is the "price" of tracking the increasing Γ/R as the filament shrinks.

7 Optional: spectrum of relaxation times (multi-mode UCM) and the general radius law

Entov & Hinch (1997) emphasize that real polymer solutions often have many relaxation modes. The UCM generalization is a sum of independent Maxwell modes:

$$\boldsymbol{\tau} = \sum_{i=1}^{N} \boldsymbol{\tau}^{(i)}, \qquad \boldsymbol{\tau}^{(i)} + \lambda_i \overset{\nabla}{\left(\boldsymbol{\tau}^{(i)}\right)} = 2\eta_i \boldsymbol{D}, \qquad \eta_i = G_i \lambda_i.$$

In the same elastocapillary large-stress limit, each mode's axial stress approximately satisfies

$$\dot{\tau}_{zz}^{(i)} \approx \left(2\dot{\varepsilon} - \frac{1}{\lambda_i}\right) \tau_{zz}^{(i)}.$$

Following the same steps as above,[9] one obtains a radius evolution of the form

$$R(t) = R_0 \left(\frac{R_0\, G(t)}{\Gamma}\right)^{1/3}, \qquad G(t) = \sum_{i=1}^{N} G_i e^{-t/\lambda_i},$$

where $G(t)$ is exactly the *linear stress-relaxation modulus* for a discrete spectrum. This is the same structure as Eq. (9)–(10) in Entov & Hinch (1997), written in modulus/relaxation-time language.

Differentiating $\ln R$ gives the strain-rate:

$$\dot{\varepsilon}(t) = -2\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{G}(t)}{G(t)}.$$

Thus the instantaneous strain-rate equals 2/3 *of the instantaneous stress-relaxation rate.* When a single mode dominates $G(t)$ at some time, $\dot{G}/G \approx -1/\lambda_{\text{dom}}$ and

$$\dot{\varepsilon} \approx \frac{2}{3\lambda_{\text{dom}}}, \qquad \text{Wi}_{\text{dom}} \approx \frac{2}{3}.$$

[9]See Ref. [1] for detailed derivation using a multi-mode Oldroyd-B/FENE-CR model.

8 What about breakup? A key limitation of UCM

The UCM model corresponds to an *infinitely extensible* elastic element. In the simplified uniform-filament theory above, that means:

- the filament radius decreases exponentially and tends to zero only as $t \to \infty$;
- there is no intrinsic finite-time "breakup" mechanism in the ideal UCM model.

Real filaments *do* break. In the Entov & Hinch (1997) paper, breakup arises because the springs have *finite extensibility* (FENE): once chains are nearly fully stretched, the constitutive behavior changes and a late-time regime appears that can lead to a finite-time pinch-off. If you want a breakup prediction, you must add physics beyond UCM: finite extensibility (FENE-P, FENE-CR), chain scission, concentration changes, surface instabilities, or inertia/solvent-dominated pinch-off at very small radii.

Summary (one paragraph)

The elastocapillary regime comes from three ingredients: (i) the kinematic condition $\dot{R} = -(\dot{\varepsilon}/2)R$, (ii) the free-surface stress boundary conditions leading to $\Gamma/R = 3\eta_s\dot{\varepsilon} + (\tau_{zz} - \tau_{rr})$, and (iii) Maxwell relaxation with upper-convected stretching, which in the large-stress limit gives $\dot{\tau}_{zz} = (2\dot{\varepsilon} - 1/\lambda)\tau_{zz}$. Combining (i)–(iii) forces $\dot{\varepsilon} = 2/(3\lambda)$ and thus $\mathrm{Wi} = 2/3$, and yields the exponential law $R(t) \propto e^{-t/(3\lambda)}$.

References

[1] V. M. Entov and E. J. Hinch, "Effect of a spectrum of relaxation times on the capillary thinning of a filament of elastic liquid," *J. Non-Newtonian Fluid Mech.* **72** (1997) 31–53.

Appendix

We now present an alternative approach to the time evolution of filament radius $R(t)$. In the elastocapillary regime, the governing equations are

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{1}{2}\dot{\varepsilon}R, \qquad \frac{\mathrm{d}\tau_{zz}}{\mathrm{d}t} = \left(2\dot{\varepsilon} - \frac{1}{\lambda}\right)\tau_{zz} + 2G\dot{\varepsilon}, \qquad \frac{\Gamma}{R} = \tau_{zz} - \tau_{rr}.$$

Since the filament and polymers are mainly stretched in the z direction, we assume $|\tau_{zz}| \gg |\tau_{rr}|$. We also specify an initial condition: $\tau_{zz}\left(t^\dagger\right) = G$. For $t > t^\dagger$, $\tau_{zz} \gg G$ (see Section 6 for more detail about these assumptions). With this, the equations can be simplified as

$$\dot{R} = -\frac{1}{2}\dot{\varepsilon}R, \qquad \dot{\tau}_{zz} = 2\dot{\varepsilon}\left(G + \tau_{zz}\right) - \frac{\tau_{zz}}{\lambda} \approx \left(2\dot{\varepsilon} - \frac{1}{\lambda}\right)\tau_{zz}, \qquad \frac{\Gamma}{R} = \tau_{zz} - \tau_{rr} \approx \tau_{zz}.$$

Assume an initial condition: $R\left(t^\dagger\right) = R_0$. Integrating the first and the second equation gives

$$R(t) = R_0 \exp\left(\int_{t^\dagger}^{t} -\frac{1}{2}\dot{\varepsilon}\,\mathrm{d}\tilde{t}\right), \qquad \tau_{zz} = G\exp\left[\int_{t^\dagger}^{t}\left(2\dot{\varepsilon} - \frac{1}{\lambda}\right)\mathrm{d}\tilde{t}\right],$$

respectively. Eliminating $\dot{\varepsilon}$ with these two equations, then combined with the third equation:

$$\tau_{zz}(t) = G\left(\frac{R_0}{R}\right)^4 \exp\left[-\frac{\left(t - t^\dagger\right)}{\lambda}\right] \quad \Rightarrow \quad \boxed{\frac{R(t)}{R_0} = \left(\frac{GR_0}{\Gamma}\right)^{\frac{1}{3}} \exp\left[-\frac{\left(t - t^\dagger\right)}{3\lambda}\right]}.$$

This result is valid for $t \geq t^\dagger$.

There is a small difference in the prefactor here from the acknowledged form, which is a missing $2^{1/3}$ in the denominator, namely

$$\left(\frac{GR_0}{\Gamma}\right)^{\frac{1}{3}} \qquad \text{versus} \qquad \left(\frac{GR_0}{2\Gamma}\right)^{\frac{1}{3}}.$$

This is due to the improper treatment of the tension in the filament [A. Gaillard *et al.*, ***J. Fluid Mech.***, 2025].